Lezione 12

8.4 $\quad \mathbb{R}^3 \qquad U = \mathrm{Span}\left(\underset{v_1}{\begin{pmatrix}1\\2\\0\end{pmatrix}}, \underset{v_2}{\begin{pmatrix}1\\0\\2\end{pmatrix}}\right) \qquad W = \{x+y+z=0\}$

Basi ortonormali per U e W e poi una base ortonormale

con $v_1 \in U \cap W$, $v_2 \in U$, $v_3 \in W$

G.S.: $\quad w_1 = v_1$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{pmatrix}1\\0\\2\end{pmatrix} - \frac{1}{5}\begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$= \begin{pmatrix}4/5\\-2/5\\2\end{pmatrix} \qquad w_1 = \begin{pmatrix}1\\2\\0\end{pmatrix}, \quad w_2 = \begin{pmatrix}4/5\\-2/5\\2\end{pmatrix}$$

$$\overline{w_1} = \frac{w_1}{\|w_1\|} = \frac{1}{\|w_1\|}\cdot\begin{pmatrix}1\\2\\0\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\\0\end{pmatrix} = \frac{\sqrt{5}}{5}\begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}\sqrt{5}/5\\ \frac{2}{5}\sqrt{5}\\ 0\end{pmatrix}$$

$$\overline{w_2} = \frac{w_2}{\|w_2\|} = \frac{1}{\|w_2\|}\begin{pmatrix}4/5\\-2/5\\2\end{pmatrix} = \frac{1}{\sqrt{\underbrace{\frac{20}{25}}_{4/5}+4}}\begin{pmatrix}4/5\\-2/5\\2\end{pmatrix} = \frac{1}{\sqrt{24/5}}\begin{pmatrix}4/5\\-2/5\\2\end{pmatrix}$$

$$= \sqrt{\frac{5}{24}}\begin{pmatrix}4/5\\-2/5\\2\end{pmatrix} = \frac{\sqrt{5\cdot 24}}{24}\begin{pmatrix}4/5\\-2/5\\2\end{pmatrix} = \frac{2\sqrt{30}}{24}\begin{pmatrix}4/5\\-2/5\\2\end{pmatrix}$$

$$= \frac{\sqrt{30}}{12}\begin{pmatrix}4/5\\-2/5\\2\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{30}}{15}\\ -\frac{\sqrt{30}}{30}\\ \frac{\sqrt{30}}{6}\end{pmatrix}$$

$$\|\overline{w}_1\| = \sqrt{\frac{30}{15^2} + \frac{30}{30^2} + \frac{30}{6^2}} = \sqrt{\frac{2}{15} + \frac{1}{30} + \frac{5}{6}}$$

$$= \sqrt{\frac{4+1+25}{30}} = 1$$

$$W = \{x+y+z=0\} \qquad v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

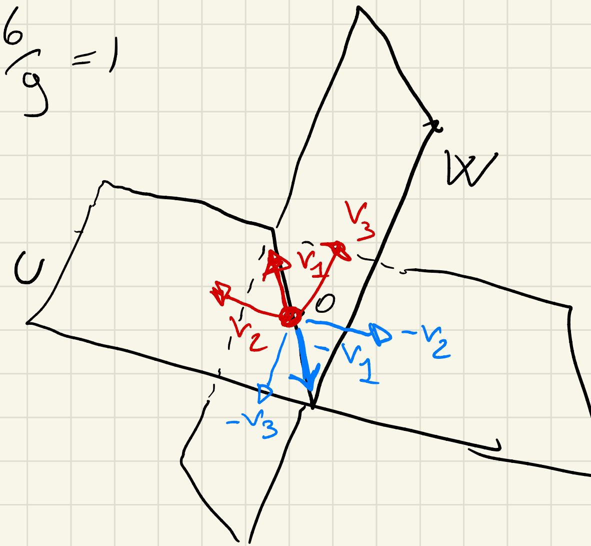
$$w_1 = v_1$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \frac{-1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} \qquad \overline{w}_2 = \begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{pmatrix}$$

$$\overline{w}_2 = \frac{1}{\sqrt{3/2}} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} = \frac{\sqrt{6}}{3} \begin{pmatrix} 1/2 \\ 1/2 \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{6}/6 \\ \sqrt{6}/6 \\ -\sqrt{6}/3 \end{pmatrix}$$

$$W = \{x+y+z=0\} \qquad \frac{1}{6} + \frac{1}{6} + \frac{6}{9} = 1$$



Vettore ortogonale a W:

v_2 $\quad u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \overset{?}{\in} \mathrm{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right) = U$

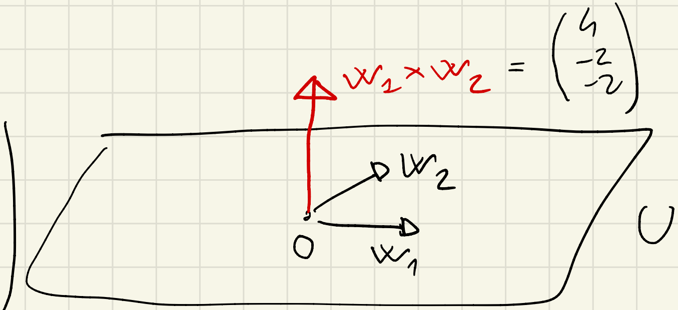
Sì, infatti $\quad u = \frac{1}{2}\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Altra strada: vettore ortogonale a U: $\quad v_2 = u$

$$\overset{w_1}{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}} \times \overset{w_2}{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}} = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} = v_3$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}$$

$$v_1 = v_2 \times v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix}$$



$$\bar{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \qquad \bar{v}_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$ da ortonormalizzare

Altra strada: Determiniamo $U \cap W$

P P
P C ←
C C ←

$$U = \text{Span}\left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right) = \left\{ \begin{pmatrix} t \\ 2t \\ 0 \end{pmatrix} + \begin{pmatrix} u \\ 0 \\ 2u \end{pmatrix} \right\} = \left\{ \begin{pmatrix} t+u \\ 2t \\ 2u \end{pmatrix} \right\}$$

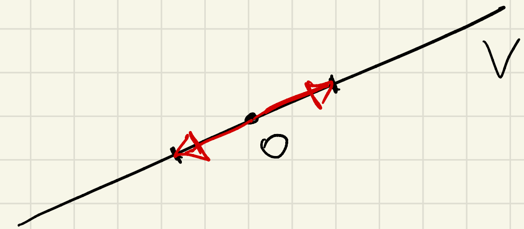
Inserisco $(t+u, 2t, 2u)$ nell'equazione di W

$W = \{x+y+z=0\}$ ottengo: $t+u+2t+2u=0$

$t+u=0$ $\qquad$ $3t+3u=0$

$u=1 \Rightarrow t=-1$

$\text{Span}\begin{pmatrix}0\\-2\\2\end{pmatrix} = U \cap W$ $\qquad$ $v_1 = \begin{pmatrix}0\\-1\\1\end{pmatrix}$



Completando a base ortogonale di U trovo v_2

" " " " " W " " v_3

8.8 $\quad U = \{x+y-2z=0\}$

Scrivi p_U e r_U

$U^\perp = \text{Span}\begin{pmatrix}1\\1\\-2\end{pmatrix}$

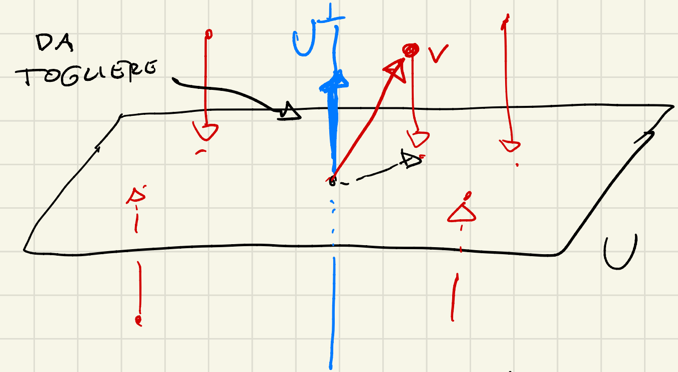
V retta

$\|v\|=1$ in V $\qquad v \in V$

$\|\lambda v\| = 1 \Rightarrow \lambda = \pm 1$

$\|\lambda v\| = |\lambda| \|v\| = |\lambda|$

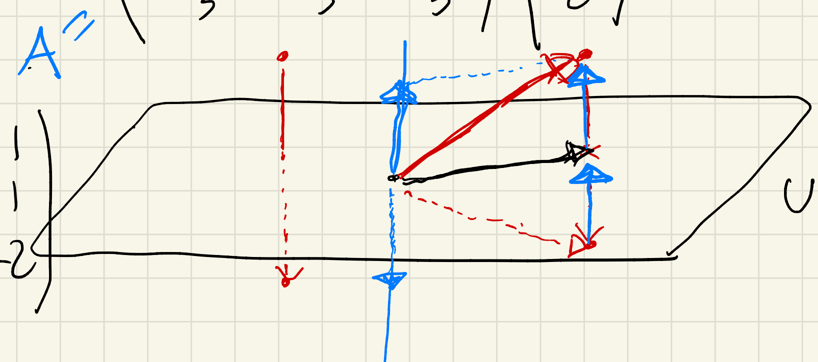
$$P_U\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - \frac{x+y-2z}{6}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$



$$= \begin{pmatrix} \frac{5}{6}x - \frac{1}{6}y + \frac{1}{3}z \\ -\frac{1}{6}x + \frac{5}{6}y + \frac{1}{3}z \\ \frac{1}{3}x + \frac{1}{3}y + \frac{1}{3}z \end{pmatrix} = \begin{pmatrix} \frac{5}{6} & -\frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{5}{6} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

RIFLESSIONE = SIMMETRIA

$$r_U\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} - 2\,\frac{x+y-2z}{6}\begin{pmatrix}1\\1\\-2\end{pmatrix}$$



$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \frac{x+y-2z}{3}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3}x - \frac{1}{3}y + \frac{2}{3}z \\ -\frac{1}{3}x \quad \frac{2}{3}y + \frac{2}{3}z \\ \frac{2}{3}x + \frac{2}{3}y - \frac{1}{3}z \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

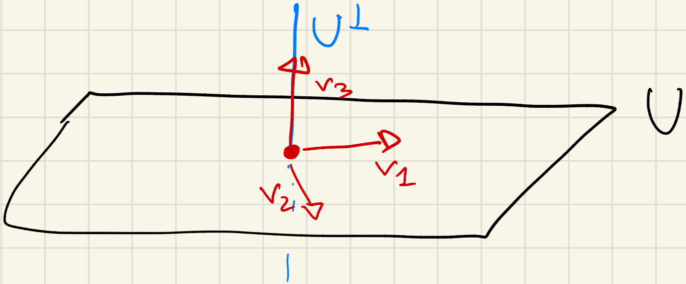
B'

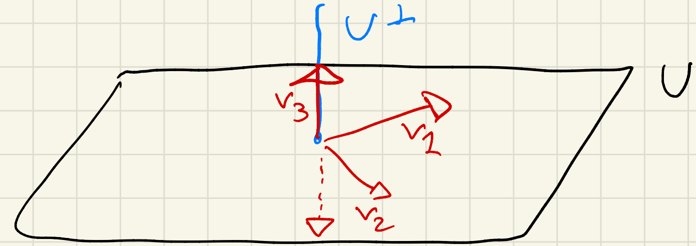
$$\boxed{\begin{aligned} &A = [p_U]_e^e \\ &\operatorname{Ker} A = U^\perp = V_0 \\ &\operatorname{Im} A = U = V_1 \end{aligned}}$$

$$B = [r_U]_e^e$$

$$U^\perp = V_{-1}$$

$$U = V_1$$





$\mathcal{B} = \{v_1, v_2, v_3\}$

$$[p_U]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

NON È INVERTIBILE

$\det = 0$

$\ker = U^\perp$

$\mathrm{Im} = U$

$$[r_U]^{\mathcal{B}}_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

INVERTIBILE

$r_U \circ r_U = id$

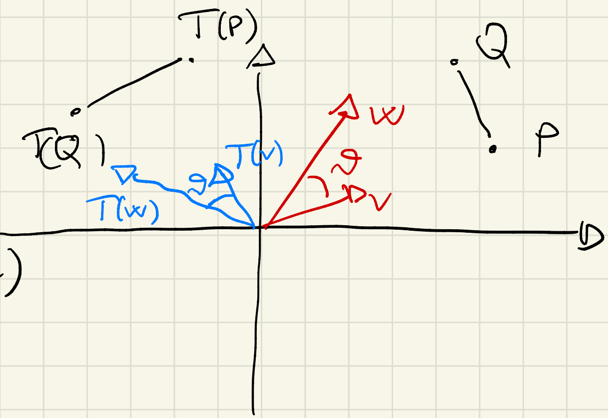
$\ker =$ banale $\{0\}$

$\mathrm{Im} =$ tutto $\mathbb{R}^3$

ISOMETRIE

Det: V sp. vett. con prod. scalare def +.

Un isomorfismo $T: V \to V$ è una ISOMETRIA VETTORIALE se
ENDOMORFISMO INVERTIBILE $\forall\, v, w \in V$ $\quad \langle v, w \rangle = \langle T(v), T(w) \rangle$



Prop: Sono fatti equivalenti per un isomorfismo $T: V \to V$:

1) T è isometria (cioè T preserva il prod. scalare)
2) T preserva la norma, cioè:
$$\|T(v)\| = \|v\| \quad \forall v \in V$$
3) T preserva la distanza, cioè
$$\forall P, Q \in V \qquad d(P, Q) = d(T(P), T(Q))$$

Conseguenza: vengono preservati gli angoli fra vettori

dim: 1) $\Rightarrow$ 2) $\quad \|T(v)\| = \sqrt{\langle T(v), T(v) \rangle} \underset{(1)}{=} \sqrt{\langle v, v \rangle} = \|v\|$

2) $\Rightarrow$ 3) $\quad P, Q \in V$

$$d(P,Q) = \|\overrightarrow{PQ}\| = \|Q-P\| \overset{(2)}{=}$$

$$= \|T(Q-P)\| \underset{\substack{T\\ \text{LINEARE}}}{=} \|T(Q)-T(P)\| = \|\overrightarrow{T(P)T(Q)}\| = d(T(P),T(Q))$$

$3) \Rightarrow 2)$ $\quad \|T(v)\| = d(0,T(v)) \underset{(2)}{=} d(0,v) = \|v\|$

$\boxed{\|w\| = d(0,w)}$ $\quad 2) \Rightarrow 1)$

$$\langle v,w\rangle = \frac{\langle v+w, v+w\rangle - \langle v,v\rangle - \langle w,w\rangle}{2}$$

$$\|v\| = \sqrt{\langle v,v\rangle}$$

$$= \frac{\|v+w\|^2 - \|v\|^2 - \|w\|^2}{2}$$

$$\underset{\substack{\|\\ \langle v,w\rangle}}{\underset{\text{TESI}}{\langle T(v),T(w)\rangle}} = \frac{\overbrace{\|T(v)+T(w)\|}^{\|T(v+w)\|}{}^2 - \|T(v)\|^2 - \|T(w)\|^2}{2} \underset{(2)}{=} \underset{\substack{\|\\ \langle v,w\rangle}}{\frac{\|v+w\|^2 - \|v\|^2 - \|w\|^2}{2}}$$

Prop: Se $\mathcal{B} = \{v_1, \dots, v_n\}$ base di V e $S = [g]_{\mathcal{B}}$

$T: V \to V$ isomorfismo $\quad A = [T]_{\mathcal{B}}^{\mathcal{B}}$

T isometria $\iff$ $\boxed{{}^tASA = S}$

$\boxed{\Leftarrow}$ Devo mostrare che se ${}^tASA = S$ allora T è un'isometria

Prop: $T: V \to V$ isomorfismo $\quad \mathcal{B} = \{v_1, \dots, v_n\}$ base per V

$\left(T \text{ isometria} \iff \langle v, w \rangle = \langle T(v), T(w) \rangle \ \forall v, w \in V \right)$

T isometria $\iff \langle v_i, v_j \rangle = \langle T(v_i), T(v_j) \rangle \ \forall v_i, v_j$

dim: $\boxed{\Rightarrow}$ ok

$\boxed{\Leftarrow}$ per linearità $\quad v = \lambda_1 v_1 + \dots + \lambda_n v_n$

$$w = \mu_1 v_1 + \dots + \mu_n v_n$$

$$\langle T(v), T(w) \rangle = \langle \lambda_1 T(v_1) + \dots + \lambda_n T(v_n), \mu_1 T(v_1) + \dots + \mu_n T(v_n) \rangle$$
$$= \sum_{i,j=1}^{n} \lambda_i \mu_j \langle T(v_i), T(v_j) \rangle$$
$$= \sum_{i,j=1}^{n} \lambda_i \mu_j \langle v_i, v_j \rangle = \langle v, w \rangle$$

Tornando a prima: voglio mostrare che

$${}^tASA = S \Rightarrow T \text{ isometria}$$

(Basta mostrare che $\langle v_i, v_j \rangle \overset{?}{=} \langle T(v_i), T(v_j) \rangle$)

Devo mostrare che $\langle T(v), T(w) \rangle = \langle v, w \rangle$

$$\langle T(v), T(w)\rangle = {}^t\underbrace{[T(v)]_{\mathcal{B}}}_{=} \cdot [g]_{\mathcal{B}} \cdot [T(w)]_{\mathcal{B}}$$

$$\boxed{[T(v)]_{\mathcal{B}} = A \cdot [v]_{\mathcal{B}}}$$

$$= {}^t(A \cdot [v]_{\mathcal{B}}) \cdot S \cdot A \cdot [w]_{\mathcal{B}}$$

$$= {}^t[v]_{\mathcal{B}} \cdot {}^tA \cdot S \cdot A \cdot [w]_{\mathcal{B}}$$

$$\langle v, w\rangle = {}^t[v]_{\mathcal{B}} \cdot S \cdot [w]_{\mathcal{B}}$$

Se ${}^tASA = S$ allora $\langle T(v), T(w)\rangle = \langle v, w\rangle$